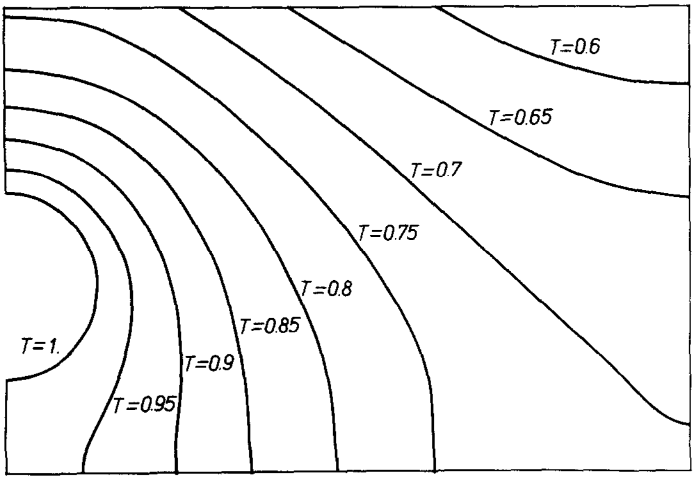

FIG. 6. Temperature distribution in the floor.

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AN ANALYTIC SOLUTION OF THE FILM THICKNESS OF LAMINAR FILM CONDENSATION ON INCLINED PIPES

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(*Received* 18 *February* 1980)

NOMENCLATURE

- r, z, ψ, coordinates of the cylindrical system [m, m, $-$];
- R, pipe radius [m];
- D, pipe diameter [m];
- H, pipe length [m];
- θ_0, angle between pipe axis and gravity directions;
- δ, film thickness [m];
- v_z, v_ψ, velocity in z, and ψ direction, respectively [m s^{-1}];
- g, acceleration of free fall [m s^{-2}];
- ρ, density of fluid [kg m^{-3}];
- η, viscosity [N s m^{-2}];
- λ, thermal conductivity [W m^{-1} K^{-1}];
- L, latent heat of condensation [J kg^{-1}];
- ΔT, difference between pipe wall and vapour temperature [K];
- $\dot{m}$, mass flow rate [kg s^{-1}];
- h, h_m, local and mean coefficient of heat transfer [W m^{-2} K^{-1}];
- C_1, C_2, constants;
- ξ, Φ, dimensionless variables, equation (6);
- τ, variable of integration.

1. INTRODUCTION

A DETAILED description of the laminar film condensation of pure saturated vapour on inclined cylinders has been given by Hassan and Jakob [1]. Applying Nusselt's classical theory of film condensation [2], they derived a partial differential equation for the film thickness. The numerical solution of this equation obtained by a finite difference method, serves as a basis for their further conclusions.

Reconsidering the problem of laminar film condensation on inclined pipes, the present author found that instead of a numerical one, an analytic solution of the partial differential equation by means of the method of characteristics can be given. This interesting result, not found in literature, will be described in this note. Some of the results of Hassan and Jakob [1] will be verified using the analytic expression of the film thickness.

2. THE BOUNDARY VALUE PROBLEM FOR THE FILM THICKNESS

Referring to Hassan and Jakob [1] for a more complete description of the problem and for a number of conditions and assumptions, a short review will be given here. We repeat that only a laminar film will be considered in the case that the film thickness is small with respect to the pipe radius.

Saturated vapour is assumed to condense at the outside of a circular cylindrical pipe. Wetting of the pipe wall causes a fluid film which moves along the pipe surface under influence of gravity. A cylindrical coordinate system (r, z, ψ) as shown in Fig. 1, is introduced. In order to determine a differential equation for the film thickness we consider the element of the fluid film defined by $[R - R + \delta, z - z + \Delta z, \psi - \psi + \Delta\psi]$, Fig. 1.

The amount of fluid condensing at the vapour side of this film element, given by

$$\Delta \dot{m} = \frac{\lambda \Delta T R}{L\delta} \Delta z \Delta\psi, \tag{1}$$

is equal to the net increase of the mass flow through the side walls of the element, being

$$\begin{aligned} \Delta \dot{m} = {} & \rho \int_R^{R+\delta(\psi, z+\Delta z)} v_z(r, \psi, z+\Delta z) r \Delta\psi \, dr \\ & - \rho \int_R^{R+\delta(\psi, z)} v_z(r, \psi, z) r \Delta\psi \, dr \\ & + \rho \int_R^{R+\delta(\psi+\Delta\psi, z)} v_\psi(r, \psi+\Delta\psi, z) \Delta z \, dr \\ & - \rho \int_R^{R+\delta(\psi, z)} v_\psi(r, \psi, z) \Delta z \, dr. \end{aligned} \tag{2}$$

From the equilibrium between the gravity and viscous forces within the fluid film, and under the assumption of the no-slip condition at the pipe wall and a vanishing shear stress in the fluid–vapour interface, the velocities occurring in equation (2) can be approximated by

$$v_z(r, \psi, z) = \frac{\rho g \cos\theta_0}{2\eta} \{2\delta(r-R) - (r-R)^2\} \tag{3}$$

and

$$v_\psi(r, \psi, z) = \frac{\rho g \sin\theta_0 \sin\psi}{2\eta} \{2\delta(r-R) - (r-R)^2\}. \tag{4}$$

Substituting equations (3) and (4) into equation (2), integrating equation (2), eliminating $\Delta\dot{m}$ from equations (1) and (2), and some straightforward calculation, result in the quasilinear differential equation of the first order for the film thickness

$$\cos\theta_0 \delta^3 \frac{\partial\delta}{\partial z} + \frac{1}{3} \frac{\sin\theta_0}{R} \delta \frac{\partial}{\partial\psi} (\sin\psi\,\delta^3) = \frac{\eta\lambda\Delta T}{L\rho^2 g}. \tag{5}$$

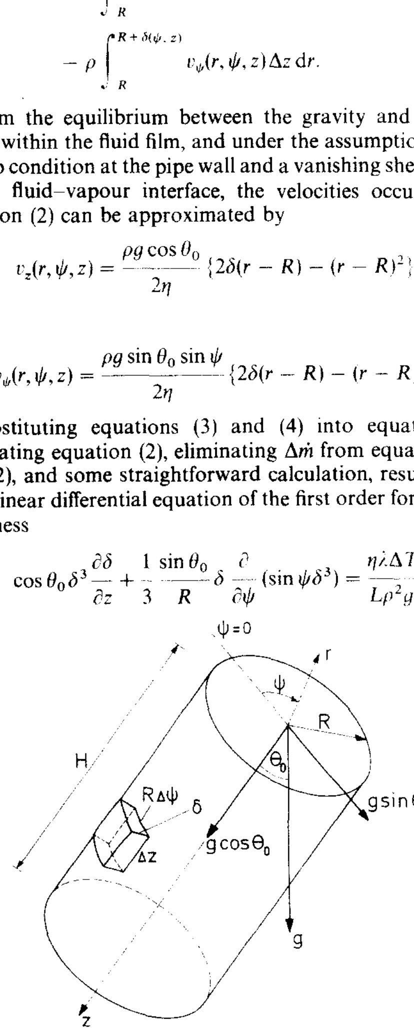

FIG. 1. The circular cylindrical pipe showing the fluid element defined by the cylindrical coordinate system.

Apart from a somewhat different form, equation (5) is similar to Hassan and Jakob's equation for the film thickness. Our problem now consists of solving equation (5) under the condition that the film thickness is equal to zero at $z = 0$ for $-\pi \leq \psi \leq \pi$.

Defining dimensionless variables by

$$\xi = \frac{z}{R} \operatorname{tg}\theta_0, \quad \Phi(\psi, \xi) = \frac{L\rho^2 g \sin\theta_0}{4\eta\lambda\Delta T R} \{\sin\psi\,\delta^3(\psi, z)\}^{4/3} \tag{6}$$

the boundary problem transforms into

$$\frac{\partial\Phi}{\partial\xi} + \sin\psi \frac{\partial\Phi}{\partial\psi} = \sin\psi^{4/3} \tag{7}$$

with the boundary condition

$$\Phi(\psi, \xi) = 0 \quad \text{for } \xi = 0 \quad \text{and} \quad -\pi \leq \psi \leq \pi. \tag{8}$$

This problem has an appropriate form to be solved analytically by the method of characteristics, compare, e.g. Courant and Hilbert [3], or Ames [4].

3. THE ANALYTIC SOLUTION OF THE FILM THICKNESS

In the following we will describe the idea of the method of solution. At each point of an integral surface being a solution of equation (7), the tangential directions are perpendicular to the normal to that integral surface defined by the directions

$$\left(\frac{\partial\Phi}{\partial\xi}, \frac{\partial\Phi}{\partial\psi}, -1\right) \tag{9}$$

Using equation (7), these characteristic, tangential directions are determined by

$$d\xi = \frac{d\psi}{\sin\psi} = \frac{d\Phi}{\sin\psi^{4/3}}. \tag{10}$$

By a simple integration of the differential equations (10), the respective sets of characteristics

$$\operatorname{tg}(\psi/2)\exp(-\xi) = C_1 \tag{11}$$

and

$$\int_0^\psi \sin\tau^{1/3} d\tau - \Phi(\psi, \xi) = C_2 \tag{12}$$

are obtained. These characteristics (11) and (12) have the property that the directions of its tangential plane at each of its points are defined by equation (10). Hence, these characteristics are integral surfaces of the original differential equation (7). The next step in the method consists of choosing that particular surface from the set (11) and (12), satisfying the boundary condition (8). Under this condition, the expressions for the constants C_1, equation (11) and C_2, equation (12), reduce to

$$C_1 = \operatorname{tg}(\psi/2) \quad \text{or} \quad \psi = 2 \operatorname{arc\,tg} C_1 \tag{13}$$

and

$$C_2 = \int_0^\psi \sin\tau^{1/3} d\tau. \tag{14}$$

Clearly, between the constants there exists the relation

$$C_2 = \int_0^{2 \operatorname{arc\,tg} C_1} \sin\tau^{1/3} d\tau. \tag{15}$$

Substituting equations (11) and (12) into equation (15) yields the analytic solution of the boundary value problem (7) and (8)

$$\int_0^\psi \sin\tau^{1/3} d\tau - \Phi(\psi, \xi) = \int_0^{2 \operatorname{arc\,tg}\{\operatorname{tg}(\psi/2)\exp(-\xi)\}} \sin\tau^{1/3} d\tau. \tag{16}$$

Transforming the various quantities of equation (16) in the original ones, equation (6), the analytic expression of the film thickness results in

$$\delta(\psi, z) = \left(\frac{4\lambda\Delta T\eta R}{\rho^2 gL\sin\theta_0}\right)^{1/4} (\sin\psi)^{-1/3} \times\left(\int_0^{\psi} \sin\tau^{1/3}[1 - \{\cos^2(\tau/2)\exp(\mathrm{tg}\,\theta_0 z/R) + \sin^2(\tau/2)\exp(-\mathrm{tg}\,\theta_0 z/R)\}^{-4/3}]\,\mathrm{d}\tau\right)^{1/4}. \quad (17)$$

APPLICATIONS

Defining the local coefficient of heat transfer by

$$h(\psi, z) = \lambda\delta^{-1}(\psi, z), \quad (18)$$

the mean coefficient of heat transfer for a circular cylindrical pipe depends on the film thickness as

$$h_m = \frac{\int_0^H\int_0^\pi h(\psi, z)R\,\mathrm{d}\psi\,\mathrm{d}z}{\int_0^H\int_0^\pi R\,\mathrm{d}\psi\,\mathrm{d}z} = \frac{\lambda}{\pi H}\int_0^H\int_0^\pi \delta^{-1}(\psi, z)\mathrm{d}\psi\,\mathrm{d}z. \quad (19)$$

Substituting equation (17), the mean coefficient of heat transfer can be calculated for every length and diameter of the pipe and any position of the pipe axis with respect to gravity.

For some special cases, the film thickness and the mean coefficient can be represented by simpler expressions. Firstly, we consider the case that under the condition that $\mathrm{tg}\,\theta_0 z/R$ is very small, the approximation

$$\exp(\mathrm{tg}\,\theta_0 z/R) \simeq 1 + \mathrm{tg}\,\theta_0 z/r \quad (20)$$

is valid. In this case the film thickness, equation (17), can be approximated by

$$\delta(\psi, z) = \left(\frac{4\lambda\Delta T\eta}{\rho^2 gL\cos\theta_0}\right)^{1/4} z^{1/4}. \quad (21)$$

The corresponding mean coefficient for heat transfer follows from the substitution of equation (21) into equation (19), leading to

$$h_m = 4/3\left(\frac{\lambda^3\rho^2 gL}{4\Delta T\eta H\cos\theta_0}\right)^{1/4}. \quad (22)$$

In the case of the vertical pipe ($\theta_0 = 0$), equations (21) and (22) result in the well-known theoretical results for a vertical plate.

A more interesting special case occurs if the second term of the integrand of the integral in equation (17) is negligible with respect to the first term. A coarse estimation of this second term in the region of integration $0 < \tau < \psi$ is given by

$$\{\cos^2(\tau/2)\exp(\mathrm{tg}\,\theta_0 z/R) + \sin^2(\tau/2) \times\exp(-\mathrm{tg}\,\theta_0 z/R)\}^{-4/3} \le \{\cos^2(\psi/2) \times\exp(\mathrm{tg}\,\theta_0 z/R)\}^{-4/3}. \quad (23)$$

The expression at the right-hand side of equation (23) can be made smaller than, say 1/10, under the assumption that

$$z > \{3/4\ln 10 - 2\ln(\cos(\psi/2))\}\,\mathrm{cotg}\,\theta_0 R. \quad (24)$$

This result shows that with the exception of a small region near $\psi = \pi$, for all values of ψ a certain characteristic length can be found, with the property that for values of z greater than this length, the film thickness can be approximated by

$$\delta(\psi, z) = \left(\frac{4\lambda\Delta T\eta R}{\rho^2 gL\sin\theta_0}\right)^{1/4} (\sin\psi)^{-1/3} \times\left[\int_0^\psi (\sin\tau)^{1/3}\mathrm{d}\tau\right]^{1/4}, \quad (25)$$

this expression being independent of z. In the region near $\psi = \pi$, the film thickness increases to very large values, approaching infinity near $\psi = \pi$.

Particularly in the case that the length of the pipe considered is large with respect to these characteristic lengths, a reasonable approximation of the mean coefficient of heat transfer can be obtained by using equation (25) as the expression for the film coefficient in equation (19), leading to

$$h_m = 0.725\left(\frac{\lambda^3\rho^2 gL}{\Delta T\eta D}\right)^{1/4} (\sin\theta_0)^{1/4}. \quad (26)$$

It is noticed that in the region near $\psi = \pi$, vanishing values of the local coefficient of heat transfer are obtained with both the exact analytic expression (17) and the approximation (25) of the film thickness. Hence, in deriving equation (26), it is justified to use equation (25) as the approximation for the film thickness also in that region.

The results of equations (25) and (26) correspond to those of Hassan and Jakob [1], being obtained in a different manner.

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